ELSEVIER
Discrete Mathematics 169 (1997) 249–255
DISCRETE MATHEMATICS

Note

A generalization of Bondy's and Fan's sufficient conditions for Hamiltonian graphs

X. Liu[a,*], B. Wei[b,1]

[a] Institute of Applied Mathematics, Academia Sinica, Beijing 100080, China
[b] Institute of Systems Science, Academia Sinica, Beijing 100080, China

Received 11 July 1994; revised 19 June 1995

Abstract

Let G be a k-connected ($k \geqslant 2$) graph on n vertices. Let S be an independent set of G. S is called essential if there exists two distinct vertices in S which have a common neighbor in G. In this paper we shall prove that if $\max\{d(u): u \in S\} \geqslant n/2$ holds for any essential independent set S with $k+1$ vertices of G, then either G is hamiltonian or G is one of three classes of exceptional graphs. This is motivated by a result of Chen et al. (1994) and generalizes the results of Bondy (1980) and Fan (1984).

1. Introduction

We consider only simple graphs $G=(V,E)$. If H and L are subsets of $V(G)$ or subgraphs of G, we denote by $N_H(L)$ the set of vertices in H which are adjacent to some vertex in L. In particular, when $H=G$ and $L=\{u\}$, we set $N_G(\{u\})=N(u)$ and $d(u)=|N(u)|$. We denote the *distance* between two vertices u and v in G by $d_G(u,v)$. An independent set S of G is called *essential*, if there exist two distinct vertices u and v in S such that $d_G(u,v)=2$. For basic graph-theoretic terminology, we refer the reader to [2].

Degree conditions have long been fundamental tools in the study of hamiltonian properties. The following classical result is due to Ore.

Theorem A (Ore [6]). *Let G be a graph on $n \geqslant 3$ vertices such that $d(u)+d(v) \geqslant n$ for any two nonadjacent vertices u, v. Then G is hamiltonian.*

* Corresponding author.
[1] Supported by National Science Fundation and post-doctoral Fundation of China.

SSDI 0012-365X(95)00323-1

Theorem A was generalized by Fan who showed that only the pair of vertices with distance 2 are essential in Theorem A.

Theorem B (Fan [5]). *Let G be a 2-connected graph on $n \geqslant 3$ vertices such that $\max\{d(u), d(v)\} \geqslant n/2$ for any two vertices u, v with $d_G(u, v) = 2$. Then G is hamiltonian.*

Using a result of Chvátal and Erdős [4] which says that if a k-connected graph has no independent set of $k+1$ vertices, then G is hamiltonian, Bondy generalized Theorem A for k-connected graphs as follows.

Theorem C (Bondy [1]). *Let G be a k-connected graph on n vertices such that $\sum_{u \in S} d(u) > (k+1)(n-1)/2$ for any independent set S with $k+1$ vertices of G. Then G is hamiltonian.*

By trying to give a common generalization of Theorems B and C, Chen et al. proved the following theorem.

Theorem D (Chen et al. [3]). *Let G be a k-connected graph on $n \geqslant 3$ vertices and $k \geqslant 2$. If $\max\{d(u): u \in S\} \geqslant n/2$ for any essential independent set S with k vertices in G, then G is hamiltonian.*

Clearly, Theorem D generalizes Theorem B, but does not generalize Theorem C, because under the conditions of Theorem C we can only get $\max\{d(u): u \in S\} \geqslant n/2$ for any independent set S with $k+1$ vertices of G.

In this paper, we shall prove the following main theorem which generalizes all theorems above.

Main Theorem. *Let G be a k-connected $(k \geqslant 2)$ graph on $n \geqslant 3$ vertices such that $\max\{d(u): u \in S\} \geqslant n/2$ for any essential independent set S with $k+1$ vertices of G. Then either G is hamiltonian or $G \in G^{(1)} \cup G^{(2)} \cup G^{(3)}$.*

Here $G^{(i)}$, $i = 1, 2, 3$ are the sets of graph which are defined as follows.

For a positive integer i, let K_i denote the complete graph on i vertices. $G^{(1)}$ denotes all spanning subgraphs of $K_1 + (K_p \cup K_q \cup K_r \cup T)$, where $p, q, r \geqslant 2$, $p + q + r = n - 1$ and T denotes the edge set of a triangle containing exactly one vertex of K_p, K_q and K_r. $G^{(2)}$ denotes all spanning subgraphs of $K_p \cup K_q \cup K_r \cup T_1 \cup T_2$, where $p, q, r \geqslant 3$, $p + q + r = n$ and T_1 and T_2 are the edge sets of two vertex-disjoint triangles each containing exactly one vertex from K_p, K_q and K_r. $G^{(3)}$ denotes all spanning subgraphs of $K_k + (\bigcup_{i=1}^{k+1} K_{n_i})$ with $k + \sum_{i=1}^{k+1} n_i = n$. The plus sign above denotes the join operation.

It is easy to see that if $G \in G^{(1)} \cup G^{(2)} \cup G^{(3)}$, then G does not satisfy the conditions of Theorems A–D. Let H be any graph on $(n-1)/2$ vertices, then the graph $H + (n+1)/2 K_1$ show that the degree bound of the Main Theorem is best possible.

2. Notations and proofs

Let C be a cycle in a k-connected graph G and with a fixed direction of traversing C. For any $u \in V(C)$ we denote by u^+ its successor and by u^- its predecessor on C. For any two vertices, a, b on C we denote by $C[a,b]$ the subpath of C from a to b (in the chosen direction). For $C[a^+,b]$ we also write $C(a,b]$, and similarly $C[a,b) = C[a,b^-]$. For some $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced by S. For a subgraph F of G, we use $\omega(F)$ to denote the number of components of F.

In this section, we use a result of Bondy and Chvátal [2], which says that G is hamiltonian if and only if $G + uv$ is hamiltonian for any $uv \notin E$ and $d(u) + d(v) \geqslant n$, to prove our main theorem.

Assume that G is a nonhamiltonian graph which satisfies the conditions of the Main Theorem and has as many edges as possible. Set $V_r = \{u\colon d(u) \geqslant n/2\}$. Then $G[V_r]$ is a complete subgraph of G. Take a longest cycle C such that C contains V_r and $\omega(G\backslash C)$ is minimum. For a component H of $G\backslash C$, we label $N_C(H) = \{x_1, x_2, \dots, x_d\}$ according to the chosen direction of C (subscripts are considered modulo d). Take $v_i \in V(H)$ such that $v_i x_i \in E$ for $1 \leqslant i \leqslant d$. Set $u_i = x_i^+$, $w_i = x_{i+1}^-$ and $T_i = V(C[u_i, w_i])$ for $1 \leqslant i \leqslant d$. Denote by $A_i(B_i)$ the set of vertices $a_i(b_i)$ such that there exists an $a_i w_i$-path ($u_i b_i$-path) in G with vertex set T_i for $1 \leqslant i \leqslant d$. Obviously, $y^- \in A_i$ if $y \in N(u_i) \cap T_i$ and $y^+ \in B_i$ if $y \in N(w_i) \cap T_i$. For $1 \leqslant i \leqslant d$, if $u_i w_i \notin E$, let $y_i \in N_{T_i}(u_i)$ such that $N(u_i) \cap V(C(y_i, w_i]) = \emptyset$ and $z_i \in N_{T_i}(w_i)$ such that $N(w_i) \cap V(C[u_i, z_i)) = \emptyset$. If $u_i w_i \in E$, let $y_i = w_i$ and $z_i = u_i$. For an essential independent set S with $k+1$ vertices, set $\Delta(S) = \max\{d(u)\colon u \in S\}$.

Since C is a longest cycle containing V_r, the following two lemmas are easy to show.

Lemma 1. (i) *If $a_i \in A_i$ and $a_j \in A_j$ $(i \neq j)$, then $N(a_i) \cap (V(H) \cup N_{G\backslash C}(a_j) \cup \{a_j\}) = \emptyset$. In particular, $\{u_1, u_2, \dots, u_d\}$ is an independent set of G.*

(ii) *If $b_i \in B_i$ and $b_j \in B_j$ $(i \neq j)$, then $N(b_i) \cap (V(H) \cup N_{G\backslash C}(b_j) \cup \{b_j\}) = \emptyset$. In particular, $\{w_1, w_2, \dots, w_d\}$ is an independent set of G.*

(iii) *Let $a_i \in A_i$ and $a_j \in A_j$ $(i < j)$. Then, for any $x \in N(a_i) \cap V(C[x_{i+1}, x_j])$, we have $a_j x^- \notin E$; and for any $x \in N(a_i) \cap V(C[x_{j+1}, x_i])$, we have $a_j x^+ \notin E$. Symmetrically, it is also true for $b_i \in B_i$ and $b_j \in B_j$ $(i \neq j)$.*

Lemma 2. *Let $a_i \in A_i$ and $b_j \in B_j$ $(i < j)$ and $a_i \neq b_j$. Then for any $x \in N(a_i) \cap V(C(x_{j+1}, x_i))$, we have $x^+ b_j \notin E$ and $x^- b_j \notin E$.*

Lemma 3. *Assume that $u_i x \in E$ and $x \in T_j$ for some $i \neq j$. If there exists some $y \in T_i$ such that $x^+ y \in E$, then for any $t \neq i$ and $1 \leqslant t \leqslant d$, we have $u_i y^+ \notin E$.*

Proof. If there exists some $t \neq i$ and $1 \leqslant t \leqslant d$ such that $u_t y^+ \in E$, then we can get a cycle $C' = x_t w_{t-1} \dots y^+ u_t u_t^+ \dots x u_i u_i^+ \dots y x^+ \dots x_i v_i P_1 v_t x_t$ when $u_t \in V(C[u_{i+1}, x))$, or $C' = x_t w_{t-1} \dots x^+ y y^- \dots u_i x x^- \dots y^+ u_t u_t^+ \dots x_i v_i P_1 v_t x_t$ when $u_t \in V(C(x^+, x_i])$, where $v_i P_1 v_t$ is a path connecting v_i and v_t in H. But in both cases, we have $V_r \subseteq V(C) \subseteq V(C')$ and $|C'| > |C|$, a contradiction. $\square$

Since G is k-connected, we have $d \geqslant k \geqslant 2$. Since V_r is a complete subgraph of G, there exists one and only one $1 \leqslant i \leqslant d$ such that $u_i \in V_r$ and $d = k$ by the assumption and Lemma 1. Without loss of generality, let $u_d \in V_r$. Symmetrically, there exists one and only one $1 \leqslant i \leqslant d$, say i, such that $w_i \in V_r$. Since $(V(G \backslash C) \cup \{u_1, u_2, \ldots, u_{d-1}\}) \cap V_r = \emptyset$ by the hypothesis of the Main Theorem and the assumption, we have $N(u_d) \cap V(G \backslash C) = \emptyset$ and $G[H]$ is a complete subgraph of G. Symmetrically, $N(w_i) \cap V(G \backslash C) = \emptyset$.

We prove our Main Theorem by the following five claims.

Claim 1. $i = d$.

In fact, if $i \neq d$, then $u_d w_i \in E$. When $w_i^- \in V_r$, then $N(w_i^-) \cap T_d \neq \emptyset$. Notice that $w_i^- w_d \notin E$ by Lemma 2, we may choose some $x \in T_d$ such that $xw_i^- \in E$ but $x^+ w_i^- \notin E$. Then $d(x^+) < n/2$. By Lemma 3, $S = \{u_1, \ldots, u_{d-1}, x^+, v_1\}$ is an essential independent set of G and $\Delta(S) < n/2$, a contradiction. Thus $w_i^- \notin V_r$. Symmetrically, we can get $u_d^+ \notin V_r$ and $u_d(u_d^+)^+ \notin E$ by Lemma 1 and the hypothesis of the Main Theorem. Then $N(u_d) \cap (T_d \backslash \{u_d^+\}) = \emptyset$, since otherwise we may choose $y \in N(u_d) \cap (T_d \backslash \{u_d^+\})$ such that $N(u_d) \cap V(C(u_d^+, y)) = \emptyset$, then $d(y^-) < n/2$ and $\{v_1, y^-, u_1, \ldots, u_{d-1}\}$ is an essential independent set with $\Delta(S) < n/2$, a contradiction. By $d(u_d) \geqslant n/2$ and $N(u_d) \cap V(G \backslash C) = \emptyset$, there must exist some vertex $x \in V(C) \backslash T_d$ such that $xu_d \in E$ and $x^+ u_d \in E$. By Lemma 1, $\{x, x^+\} \in T_j \backslash \{u_j\}$ for some $j \neq d$. Thus by Lemma 3, $S = \{v_1, u_1, \ldots, u_{d-1}, u_d^+\}$ is an essential independent set and $\Delta(S) < n/2$, a contradiction.

Claim 2. *$G[V(C[u_i, y_i))]$ is a complete subgraph of G for any $1 \leqslant i \leqslant d$.*

Otherwise, if $N(u_i) \cap V(C(u_i, y_i]) \neq V(C(u_i, y_i])$, then, since $u_i y_i \in E$, we can get some $a_i \in A_i$ such that $a_i u_i \notin E$. By Lemma 1, $\{a_i, u_i\} \cap (V(H) \cup N(u_j)) = \emptyset$ for any $j \neq i$ and $a_i \notin V_r$. Thus, $S = \{v_1, a_i, u_1, \ldots, u_{d-1}\}$ is an essential independent set of G and $\Delta(S) < n/2$, a contradiction. But if $N(u_i) \cap V(C(u_i, y_i]) = V(C(u_i, y_i])$, then there exist some $a \neq a' \in A_i$ such that $d_G(a, a') = 2$. Since $G[V_r]$ is complete, we may assume that $a \notin V_r$. When $i = d$, then $S = \{v_1, a, u_1, \ldots, u_{d-1}\}$ is an essential independent set by Lemma 1 and $\Delta(S) < n/2$, a contradiction. When $i \neq d$, also by Lemma 1, $S = \{v_1, a, a'\} \cup (\{u_1, \ldots, u_{d-1}\} \backslash \{u_i\})$ is an essential independent set and $\Delta(S) < n/2$, a contradiction.

Symmetrically, we may prove that $G[V(C(z_i, w_i]))]$ is a complete subgraph of $G\,(1 \leqslant i \leqslant d)$. Thus if $u_t w_t \in E$ for some $1 \leqslant t \leqslant d$, then $G[T_t]$ is a complete subgraph of G. In particular, $G[T_d]$ is a complete subgraph of G. By the hypothesis of the Main Theorem and Lemma 1, $T_d \subseteq V_r$.

Claim 3. $\omega(G\setminus C) = 1$.

Otherwise, let H' be a component of $G\setminus C$ and $H' \neq H$. If $N_C(H')\cap \{u_1, u_2, \dots, u_{d-1}\} = \emptyset$, then $S = \{v', v_1, u_1, \dots, u_{d-1}\}$ is an essential independent set for any $v' \in V(H')$ and $\Delta(S) < n/2$, a contradiction. If $N_C(H')\cap\{u_1, u_2, \dots, u_{d-1}\} \neq \emptyset$, then, without loss of generality, let $u_1v' \in E$, where $v' \in V(H')$. Since G is k-connected and $k = d$, for any $j \neq t$ we may choose two distinct neighbors of x_j and x_t in H, when $|H| \geqslant 2$. Thus in this subcase, we have $S = \{v', v_2, u_1^+, u_2, u_3, \dots, u_{d-1}\}$ is an essential independent set by the maximality of C and $\Delta(S) < n/2$, a contradiction. When $|H| = 1$, then by the choice of C, we also have $S = \{v', v_2, u_1^+, u_2, u_3, \dots, u_{d-1}\}$ is an essential independent set and $\Delta(S) < n/2$, a contradiction.

Claim 4. $|T_d| \geqslant 2$ *and if* $|T_d| = 2$, *then* $G \in G^{(3)}$.

If $|T_d| = 1$, which means $u_d = w_d$, then by $u_d \in V_r$ and $N(u_d)\cap V(G\setminus C) = \emptyset$, there exists some $x \in V(C[x_1, x_d))$ such that $u_dx, u_dx^+ \in E$. Thus we can easily get a cycle which contains all vertices of V_r and is longer than C, a contradiction.

If $|T_d| = 2$, since $(N(u_d)\cup N(w_d))\cap V(G\setminus C) = \emptyset$, we have $(N_C^+(u_d)\cup N_C^-(u_d))\cap N(w_d)\cap(V(C)\setminus T_d) = \emptyset$ by Lemma 2. Thus, by the hypothesis of the Main Theorem and the maximality of C, we can get $|T_i| = 1$ for any $1 \leqslant i \leqslant d-1$ and $k = (n-2)/2$. Hence $G \in G^{(3)}$.

Claim 5. *If* $d \geqslant 3$, *then* $G \in G^{(3)}$.

If $|T_d| = 2$, then Claim 5 holds by Claim 4. Hence, we only need to consider $|T_d| \geqslant 3$.

By Claims 2 and 3, for any $a \neq a' \in V(C[u_j, y_j))$, $aa' \in E$, where $1 \leqslant j \leqslant d$ and $\omega(G\setminus C) = 1$.

If there exists some $x \in T_d\setminus\{u_d, w_d\}$ such that $N(x)\cap T_i \neq \emptyset$ for some $1 \leqslant i < d$, let $xy \in E$ and $y \in T_i$. Since $G[T_d]$ is a complete subgraph of G, we have $x \in A_d\cap B_d$ and $y \notin \{u_i, w_i\}$. Thus, by Lemmas 1 and 2, $S = \{v_1, y^+, u_1, \dots, u_{d-1}\}$ is an essential independent set with $\Delta(S) < n/2$, a contradiction.

Hence, for any vertex $x \in T_d\setminus\{u_d, w_d\}$, $N_C(x)\cap(V(C)\setminus(N_C(H)\cup T_d)) = \emptyset$. We now distinguish the following two cases.

Case 1. There exist some $1 \leqslant i < j \leqslant d$ and $i \neq d-1$ such that $u_jw_i \in E$.

Case 1.1. $j \neq d$.

By Lemmas 1(iii) and Lemma 2, $x_{i+1} \notin N_C(T_d)$. Since G is k-connected, $N_C(\{u_d, w_d\})\setminus(N_C(H)\cup T_d) \neq \emptyset$. Without loss of generality, let $x \in N(u_d)\cap T_t$ and $t < d$, then $d(x^+) < n/2$ by Lemma 2 and $G[V_r]$ is complete.

If $N(x_d)\cap(T_d\setminus\{u_d\}) \neq \emptyset$, then let $a_d \in N(x_d)\cap(T_d\setminus\{u_d\})$. Since $G[V_r]$ is a complete subgraph of G, $x \neq w_i$ and $x^+u_p \notin E$ for all $1 \leqslant p \leqslant t$ by Lemma 1. When there exists some $t < p < d$ such that $u_p x^+ \in E$, we can get a $C' = x_p w_{p-1} \dots x^+ u_p u_p^+ \dots x_d a_d P_1 u_d x x^- \dots x_1 v_1 P_2 v_p x_p$, where $a_d P_1 u_d$ is a path with vertex set T_d and $v_1 P_2 v_p$ is a path connecting v_1 and v_p in H. Clearly, $C' \neq C$ and $V(C) \subseteq V(C')$, a contradiction. When $u_p x^+ \notin E$ for any $t < p < d$, then by Lemma 1, $S = \{v_1, x^+, u_1, u_2, \dots, u_{d-1}\}$ is an essential independent set and $\Delta(S) < n/2$, a contradiction.

If $N(x_d) \cap (T_d \backslash \{u_d\}) = \emptyset$, then $N(w_d) \cap (V(C) \backslash (N_C(H) \cup T_d)) \neq 0$, for G is k-connected. For the same reason as above, $N(x_1) \cap (T_d \backslash \{w_d\}) = \emptyset$. Thus, $N(T_d \backslash \{u_d, w_d\}) \subseteq \{u_d, w_d\} \cup (\{x_2, x_3, \dots, x_{d-1}\} \backslash \{x_{i+1}\})$, contrary to the fact that G is k-connected.

Case 1.2. $j = d$. Since $i \neq d-1$, by Lemma 1 $\{x_{i+1}, x_d\} \cap N(T_d \backslash \{u_d\}) = \emptyset$. Since G is k-connected, we have $N(w_d) \cap (V(C) \backslash (N_C(H) \cup T_d)) \neq \emptyset$. Thus, as in Case 1.1, we can obtain a contradiction.

Case 2. $w_i u_j \notin E$ for any $1 \leqslant i < j \leqslant d$ and $i \neq d-1$.

If there exists some $1 \leqslant i \leqslant d-1$ such that $u_i w_i \notin E$, then by Case 1, $S = \{w_1, \dots, w_i, u_i, \dots, u_{d-1}, v_i\}$ is an essential independent set and $\Delta(S) < n/2$, a contradiction.

Hence, $u_i w_i \in E$ for any $1 \leqslant i \leqslant d-1$. By Claim 2, $G[T_i]$ is a complete subgraph of G for any $1 \leqslant i \leqslant d-1$.

Case 2.1. There exist some $1 \leqslant i < j \leqslant d$ such that $u_i w_j \in E$. Let $i_0 = \min\{i: u_i w_j \in E$ for some $1 \leqslant i < j \leqslant d\}$.

If $i_0 \neq 1$ or $i_0 = 1$ but $j \neq d$, then $|T_t| \leqslant 2$ for any $i_0 \leqslant t \leqslant j-1$ by the proof of Case 1. By Lemmas 1 and 2, $x_{i_0} w_{i_0} \notin E$, $x_j w_{i_0} \notin E$ and $N(w_{i_0}) \cap (T_t \backslash \{u_t\}) = \emptyset$ for any $1 \leqslant t \leqslant d$. By the choice of i_0 and Case 1, $u_t w_{i_0} \notin E$ for any $t \neq i_0$. Thus $d(w_{i_0}) \leqslant k-1$, a contradiction.

If $i_0 = 1$ and $j = d$, then $|T_t| = 2$ for any $2 \leqslant t \leqslant d-1$. As in Case 1, we can get $w_d u_i \notin E$ for $2 \leqslant t \leqslant d-1$. Since $G[T_1]$ and $G[T_d]$ are complete subgraphs of G, by Lemma 1 and the preceding subcases, we can get $N(T_1 \cup T_d \cup \{x_1\}) \subseteq N_C(H) \cup V(H)$, when $w_{d-1} u_d \notin E$. Thus $N_C(H) \backslash \{x_1\}$ is a cut vertex set of G by $d = k \geqslant 3$, contrary to the fact that G is k-connected. When $w_{d-1} u_d \in E$, then $\{x_1, x_d\} \cap N(w_1) = \emptyset$. Thus $d(w_1) \leqslant k-1$, a contradiction.

Case 2.2. $u_i w_j \notin E$ for any $1 \leqslant i < j \leqslant d$. If $w_{d-1} u_d \notin E$, then by Case 1, $G \in G^{(3)}$. If $w_{d-1} u_d \in E$, then $|T_t| = 2$ for any $1 \leqslant t \leqslant d-2$. For the same reason as in Case 2.1, we can get $N(T_{d-1} \cup T_d \cup \{x_d\}) \subseteq N_C(H) \cup V(H)$. Thus, $N_C(H) \backslash \{x_d\}$ is a cut vertex set of G, a contradiction.

Now, by Claims 4 and 5, we need only to show that if $d = k = 2$, then $G \in G^{(1)} \cup G^{(2)}$.

Since V_r is a complete subgraph of G, $\{u_1, w_1\} \cap V_r = \emptyset$ by Lemma 1. If $u_1 w_1 \notin E$, then $S = \{v_1, u_1, w_1\}$ is an essential independent set and $\Delta(S) < n/2$, a contradiction. If $v_1 w_1 \in E$, then by Claim 2, $G[T_1]$ is a complete subgraph of G. Thus $G \in G^{(1)} \cup G^{(2)}$.

Therefore, the proof is complete.

Acknowledgements

The authors thank the referees for many valuable comments.

References

[1] J.A. Bondy, Longest paths and cycles in graphs of high degree, Research Report CORR80-16, Univ. of Waterloo, Ontario, 1980.
[2] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, London and Elsevier, New York, 1976).
[3] G.T. Chen, Y. Egawa, X. Liu and A. Saito, Essential independent set and hamiltonian cycles, J.G.T. 21 (1996) 243–250.
[4] V. Chvátal and P. Erdős, A note on hamiltonian circuits. Discrete Math. 2 (1972) 111–136.
[5] G. Fan, New sufficient conditions for cycles in graphs, J. Combin. Thoery Ser B 37 (1984) 221–227.
[6] O. Ore, A note on hamiltonian circiuts, Amer. Math. Montly 67 (1960) 55.